CLASSIFYING *-HOMOMORPHISMS

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INTRODUCTION

Theorem ("Many hands")

C^*-algebras that are unital, simple, separable, nuclear, regular, and satisfy the UCT, are classified by K-theory and traces.

- C^*-analog of the classification of injective von Neumann factors: Murray-von Neumann, Connes, Haagerup.
- No traces: Kirchberg-Phillips (1990s).
- We focus only on the case $T(A) \neq \varnothing$.
- Classifying invariant:

$$\mathrm{Ell}(A) := \Big(K_0(A),\ K_0(A)_+,\ [1_A]_0,\ K_1(A),\\ T(A),\ T(A) \times K_0(A) \to \mathbb{R}\Big)$$

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Setup: G, a countable discrete group, acting on X, a compact metric space.

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- ...in the UCT class, if G is amenable (Tu);

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- . . . regular, if X is finite dimensional and G is f.g. nilpotent.

Finite nuclear dimension (Winter-Zacharias)

- $\dim_{\mathrm{nuc}} A$: noncommutative analog of covering dimension
- $\dim_{\mathrm{nuc}} C(X) = \dim X$
- range of $\mathrm{Ell}(-)$ exhausted by C^*-algebras with $\dim_{\mathrm{nuc}} < \infty$

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$\mathcal{Z}$-stability: $A \cong A \otimes \mathcal{Z}$

- Jiang-Su algebra $\mathcal{Z}$: ∞-dim'l analog of $\mathbb{C}$.
- $\mathcal{Z} \sim_{KK} \mathbb{C}$; has unique trace.
- $\mathrm{Ell}(A) = \mathrm{Ell}(A \otimes \mathcal{Z})$

Theorem (Castillejos-Evington-Tikuisis-White-Winter)

For A is a unital, simple, separable, nuclear, nonelementary,

$$\dim_{\mathrm{nuc}} A < \infty \quad \Leftrightarrow \quad A \cong A \otimes \mathcal{Z}.$$

Comments:

- "$\Rightarrow$" is due to Winter
- Can remove "unital" from statement: Castillejos-Evington, Tikuisis.
- Conjecturally equivalent to a third condition: *strict comparison.*
- Proof developed important technique for handling complicated trace spaces.

THE ORIGINAL ROAD TO CLASSIFICATION ($\sim$2017)

Impossible to summarize decades of work in a slide.

Some recent components:

Classification of “model” algebras

- Gong-Lin-Niu ’15: classified C^*-algebras with a certain internal tracial approximation structure.
- The class exhausts range of $\mathrm{Ell}(-)$.

Realizing the approximations

- Elliott-Gong-Lin-Niu ’15: abstract conditions on a C^*-algebra $\Rightarrow$ concrete tracial approximations of GLN.
- Tikuisis-White-Winter ’17: the abstract conditions are the ones stated in the classification theorem.

We develop an alternate route to classification: beginning with von Neumann algebraic techniques inspired by work of Connes and Haagerup, we extend the KK-theoretic techniques recently developed by Schafhauser to prove classification theorems in an abstract setting.

A DIFFERENT APPROACH

We develop an alternate route to classification: beginning with von Neumann algebraic techniques inspired by work of Connes and Haagerup, we extend the KK-theoretic techniques recently developed by Schafhauser to prove classification theorems in an abstract setting.

We will (mostly) ignore the difficulties that arise from non-separability or the lack of a unit.

CLASSIFYING MORPHISMS & ALGEBRAS

Existence and uniqueness into II_1 factors

Theorem (Connes '76)

An injective von Neumann algebra is hyperfinite.

Corollary

A: nuclear C^*-algebra; M: II_1 factor.

1. *(existence)*
   $\tau \in T(A) \implies \exists *\text{-hom } \varphi\colon A \to M \text{ s.t. } \tau_M \circ \varphi = \tau.$
2. *(uniqueness)*
   $\varphi, \psi\colon A \to M\ *\text{-hom's s.t. } \tau_M \circ \varphi = \tau_M \circ \psi$
   $\implies \varphi \approx_u \psi \quad (\text{in } \|\cdot\|_2)$

Rough scheme

Produce invariant $\mathrm{inv}(-)$ s.t.
(with abstract hypotheses on A, B):

- (existence)
  $\alpha\colon\ \mathrm{inv}(A) \to \mathrm{inv}(B) \implies \exists\, \varphi\colon A \to B$ s.t. $\mathrm{inv}(\varphi) = \alpha$;
- (uniqueness)
  $\varphi, \psi\colon A \to B$ and $\mathrm{inv}(\varphi) = \mathrm{inv}(\varphi) \implies \varphi \approx_u \psi$.

Intertwining: $\mathrm{inv}(A) \cong \mathrm{inv}(B) \implies A \cong B$.

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Intertwining: $\mathrm{inv}(A) \cong \mathrm{inv}(B) \implies A \cong B$.

$\mathrm{inv}(-)$ will be more complicated than $\mathrm{Ell}(-)$.

Also want:

$$\mathrm{Ell}(A) \cong \mathrm{Ell}(B) \ \text{ yields } \ \mathrm{inv}(A) \cong \mathrm{inv}(B).$$

Existence and uniqueness for morphisms of AF algebras

A, B = (unital) AF algebras.

1. $\alpha\colon \big(K_0(A), K_0(A)_+, [1_A]_0\big) \to \big(K_0(B), K_0(B)_+, [1_B]_0\big)$
   $\implies \exists$ $*$-hom. $\varphi\colon A \to B$ s.t. $\alpha = K_0(\varphi)$.
2. $\varphi, \psi\colon A \to B$ and $K_0(\varphi) = K_0(\psi) \implies \varphi \approx_u \psi$.

Classification of AF Algebras

If $\exists$ isomorphism

$$\alpha\colon \big(K_0(A), K_0(A)_+, [1_A]\big) \to \big(K_0(B), K_0(B)_+, [1_B]_0\big),$$

then $\exists$ isomorphism $\varphi\colon A \to B$ s.t. $K_0(\varphi) = \alpha$.

ONE INGREDIENT OF INV(−): "ALGEBRAIC" K_1

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Definition

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$CU^\infty(A)$ is the *closure* of the commutator subgroup of $U^\infty(A)$.

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$CU^\infty(A)$ is the *closure* of the commutator subgroup of $U^\infty(A)$.

$\overline{K}_1^{\mathrm{alg}}(A)$ came up in Thomsen’s work on the role of the relationship between and K-theory and traces in classification theory.

Definition

$\underline{K}(A) = \bigoplus_{n=0}^{\infty} K_0(A; \mathbb{Z}/n\mathbb{Z}) \oplus K_1(A; \mathbb{Z}/n\mathbb{Z})$

Can think of $K_i(A; \mathbb{Z}/n\mathbb{Z})$ as $K_i(A \otimes \mathcal{O}_{n+1})$.

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Slogan

Can check “closeness” of $KK(\varphi)$ and $KK(\psi)$ by checking that $\underline{K}(\varphi)$ and $\underline{K}(\psi)$ agree on large finite subsets of $\underline{K}(A)$.

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A *compatible triple* $(\underline{\alpha},\beta,\gamma)\colon\ \mathrm{inv}(A)\to\mathrm{inv}(E)$ consists of

$$\underline{\alpha}\colon \underline{K}(A)\to\underline{K}(E),\quad \beta\colon \overline{K}_1^{\mathrm{alg}}(A)\to\overline{K}_1^{\mathrm{alg}}(E),\quad \gamma\colon\ \mathrm{Aff}\,T(A)\to\mathrm{Aff}\,T(E)$$

such that

$$\begin{tikzcd}
K_0(A) \arrow[r, "\rho_A"] \arrow[d, "\alpha_0"] & \mathrm{Aff}\,T(A) \arrow[r, "\mathrm{Th}_A"] \arrow[d, "\gamma"] & \overline{K}_1^{\mathrm{alg}}(A) \arrow[r] \arrow[d, "\beta"] & K_1(A) \arrow[d, "\alpha_1"] \\
K_0(E) \arrow[r, "\rho_E"] & \mathrm{Aff}\,T(E) \arrow[r, "\mathrm{Th}_E"] & \overline{K}_1^{\mathrm{alg}}(E) \arrow[r] & K_1(E)
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commutes.

$(\underline{\alpha}, \beta, \gamma)$ is *faithful and amenable on traces* if $\gamma^*\big(T(E)\big) \subseteq T(A)$ consists of faithful amenable traces.

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Theorem (C-Gabe-Schafhauser-Tikuisis-White)

- *A : sep., exact, UCT*
- *B : sep., $\mathcal{Z}$-stable, strict comparison, $T(B) \neq \varnothing$ & compact*
- $(\underline{\alpha}, \beta, \gamma)\colon \mathrm{inv}(A) \to \mathrm{inv}(B_\infty)$: *compatible triple that is faithful and amenable on traces*

Then:

- *$\exists$ full† nuclear $*$-hom. $\varphi\colon A \to B_\infty$ s.t. $\mathrm{inv}(\varphi) = (\underline{\alpha}, \beta, \gamma)$;*
- *this φ is unique up to unitary equivalence.*

APPROXIMATE CLASSIFICATION OF MORPHISMS

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[†] : $\varphi(a)$ generates B_∞ as an ideal $\forall a \neq 0$.

STRATEGY

Recall: $B_\infty := \prod_{n=1}^\infty B / \sum_{n=1}^\infty B$.

The *trace-kernel ideal* is

$$J_B := \left\{(x_n) \in B_\infty : \lim_{n\to\infty} \|x_n\|_{2,u} = 0\right\},$$

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$\rightsquigarrow$ B^∞ behaves like a finite vNa (sort of)

APPROXIMATE CLASSIFICATION OF MORPHISMS: MAJOR STEPS

$$A$$

$$0 \longrightarrow J_B \longrightarrow B_\infty \xrightarrow{q} B^\infty \longrightarrow 0$$

$$
\begin{tikzcd}
 & & A \arrow[dr, bend left, "1"] & & \\
0 \arrow[r] & J_B \arrow[r] & B_\infty \arrow[r, "q"] & B^\infty \arrow[r] & 0
\end{tikzcd}
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1. Classify morphisms into B^∞
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APPROXIMATE CLASSIFICATION OF MORPHISMS: MAJOR STEPS

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2. Classify lifts of morphisms to B_∞
   (Ext, KK techniques)

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1. Classify morphisms into B^∞
   (von Neumann techniques)
2. Classify lifts of morphisms to B_∞
   (Ext, KK techniques)
3. Adjust the K-theory, exploiting J_B

Setup: A: exact; B: $\mathcal{Z}$-stable, strict comp., $T(B) \neq \varnothing$ & compact

Theorem (Existence for lifts; CGSTW)

$\theta\colon A \to B^\infty$ full nuclear $*$-hom;

$\kappa \in KK_{\mathrm{nuc}}(A, B_\infty), \quad [q]_{KK_{\mathrm{nuc}}}\kappa = [\theta]_{KK_{\mathrm{nuc}}}$

$\Longrightarrow \exists$ *full nuclear lift* $\varphi\colon A \to B_\infty$ *of* θ *s.t.* $[\varphi]_{KK_{\mathrm{nuc}}} = \kappa$.

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(Very) roughly:

- θ determines a pullback extension e_θ whose class in $\mathrm{Ext}_{\mathrm{nuc}}(A, J_B)$ vanishes.

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- $[e_\theta] = 0 \implies e_\theta \oplus$(trivial extension) $\approx$ a split extension.
- Weyl-von Neumann-Voiculescu type absorption theorems
  $\implies e_\theta \oplus$(trivial extension) $\approx e_\theta$.
  $\implies e_\theta$ splits, and θ lifts.

What if we have two lifts φ and ψ of θ?

$$\begin{tikzcd} & & A \arrow[d, shift right, "\psi"'] \arrow[d, shift left, "\varphi"] \arrow[dr, bend left, "\theta"] & & \\ 0 \arrow[r] & J_B \arrow[r] & B_\infty \arrow[r] & B^\infty \arrow[r] & 0 \end{tikzcd}$$

Want to guarantee (a strong form of) uniqueness with a condition that can be verified by comparing invariants.

Think of Voiculescu's Theorem:

$$\begin{tikzcd}
& & A \arrow[d, shift right, "\psi"'] \arrow[d, shift left, "\varphi"] \arrow[dr, bend left, "\theta"] & & \\
0 \arrow[r] & \mathcal{K} \arrow[r] & \mathcal{B}(\mathcal{H}) \arrow[r] & \mathcal{Q}(\mathcal{H}) \arrow[r] & 0
\end{tikzcd}$$

If φ, ψ are "admissible" (faithful, nondegenerate, and $\varphi(A) \cap \mathcal{K} = \{0\} = \psi(A) \cap \mathcal{K}$), then $\varphi \approx_u \psi$.

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More can be said:

Theorem (Dadarlat-Eilers '01)

Suppose: $\varphi, \psi \colon A \to \mathcal{B}(\mathcal{H})$ *are admissible lifts of* θ.

Then:

$[\varphi, \psi] = 0 \in KK(A, \mathcal{K}) \implies \varphi \approx_u \psi$ *via unitaries in* $\mathcal{K} + \mathbb{C}1_{\mathcal{H}}$.

$$\begin{tikzcd} & & A \arrow[d, shift right, "\psi"'] \arrow[d, shift left, "\varphi"] \arrow[dr, bend left, "\theta"] & & \\ 0 \arrow[r] & J_B \arrow[r] & B_\infty \arrow[r] & B^\infty \arrow[r] & 0 \end{tikzcd}$$

Theorem (Uniqueness for lifts; CGSTW)

A: exact;

B: $\mathcal{Z}$-stable, strict comparison, $T(B) \neq \varnothing$ & compact;

φ, ψ: full nuclear lifts of θ.

$$[\varphi, \psi] = 0 \in KL_{\mathrm{nuc}}(A, J_B) \quad \Longrightarrow \quad \varphi \approx_u \psi \text{ via unitaries in } \widetilde{J_B}\,.$$

SUMMARY

Using inv(—) can provide $\exists$ and ! for $*$-hom's

$$\boxed{\begin{array}{c}\text{sep., simple,}\\ \text{nuclear, UCT}\end{array}} \longrightarrow \boxed{\begin{array}{c}\text{sep., simple, nuclear,}\\ \text{finite, } \mathcal{Z}\text{-stable}\end{array}}$$

Using inv(—) can provide $\exists$ and ! for $*$-hom's

$$\boxed{\begin{array}{c}\text{sep., simple,}\\ \text{nuclear, UCT}\end{array}} \longrightarrow \boxed{\begin{array}{c}\text{sep., simple, nuclear,}\\ \text{finite, } \mathcal{Z}\text{-stable}\end{array}}$$

Can use this to classify algebras ... even in the non-unital setting.

Theorem (CGSTW)

Suppose A and B are non-unital, simple, separable, nuclear, $\mathcal{Z}$-stable C^*-algebras satisfying the UCT.

Any isomorphism $\mathrm{Ell}^+(A) \xrightarrow{\simeq} \mathrm{Ell}^+(B)$ lifts to an isomorphism $A \xrightarrow{\simeq} B$.

Thank you!